Title	REPRESENTATION OF FINITE GROUPS AND THE FIRST BETTI NUMBER OF BRANCHED COVERINGS OF A UNIVERSAL BORROMEAN ORBIFOLD (Hyperbolic Spaces and Discrete Groups II)
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ボロミアン普遍オービフォルドの分岐被覆のベッチ数と有限群の表現 (REPRESENTATION OF FINITE GROUPS AND THE FIRST BETTI NUMBER OF BRANCHED COVERINGS OF A UNIVERSAL BORROMEAN ORBIFOLD)

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1. Motivation. The main object is the first homology of regular branched coverings of a hyperbolic 3-orbifold. We shall stick to a single, but universal, example of 3-orbifolds, which is called $B_{4,4,4}\backslash\mathbb{H}^3$ by Hilden, Lozano and Montesinos[HLM1]. The homology is given a structure of $\mathbb{C}[G]$-module by the action of covering transformation group G. The main result is the structure of the $\mathbb{C}[G]$-module. The investigation is motivated by the following problem in 3-dimensional topology:

Problem. *Does every aspherical 3-manifold have a finite-sheeted cover of positive first Betti number?*

This problem was raised by Thurston, which can be one of the crucial steps towards his hyperbolization conjecture of irreducible atoroidal 3-manifolds through his hyperbolization theorem for Haken 3-manifolds. The lemma below illustrate how irreducible components of the $\mathbb{C}[G]$-module is related to the first Betti numbers of unbranched coverings of a given 3-manifold.

Lemma. *Suppose that Γ is an orientation-preserving cocompact Kleinian group and Γ_0 a normal subgroup of finite index in Γ. Then we have*

$$H_*(\Gamma\backslash\mathbb{H}^3,\mathbb{C}) \simeq H_*(\Gamma_0\backslash\mathbb{H}^3,\mathbb{C})^{\Gamma/\Gamma_0}$$

where superscript Γ/Γ_0 denotes the fixed point set by the action of Γ/Γ_0.

The proof is a direct application of the basic homology theory , in particular the transfer map.

Now let us recall the definition of universal groups.

Definition. *Kleinian group Γ is universal if, for any given closed 3-manifold M, there is subgroup Γ_M of finite index in Γ such that $\Gamma_M\backslash\mathbb{H}^3$ is homeomorphic to M.*

See [HLM2] for the universality of Kleinian group $B_{4,4,4}$.

We denote by T_Γ the subgroup of Γ generated by all elements of finite order in Γ. The following assersion easily follows from above Lemma.

Proposition. *For given closed 3-manifold M, any subgroup Γ_M of universal group $B_{4,4,4}$ associated to M in the definition and each normal subgroup Γ_0 of finite index in $B_{4,4,4}$, we can find a finite-sheeted (unbranched) covering $\tilde{M}_{\Gamma_0}$ of M with*

$$b_1(\tilde{M}_{\Gamma_0}) \geq \ dim(H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})^{T_{\Gamma_M}\Gamma_0/\Gamma_0})$$

where $b_1(\cdot)$ denotes the first Betti number.

Hence the information of the irreducible component of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$ gives us the lower bound of the betti number of 3-manifolds which is covered by $\Gamma_0\backslash\mathbb{H}^3$, possibly with branches. In view of the proposition Thurston's problem can be devided into two parts, the first is the investigation of the irreducible component of G-module $dim(H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C}))$ for various Γ_0 and the second is finding the nice Γ_0 in which the images of T_{Γ_M} is 'small'. We shall investigate the first part.

2. Results. $B_{4,4,4}$ is normalized by mutually orthogonal hyperbolic reflections r_1, r_2 and r_3. r denotes orientation reversing element r_1 or $r_1r_2r_3$ of the normalizer.

Theorem A. *Let Γ_0 be the r-normal subgroup of $B_{4,4,4}$ with finite index. If the irreducible representation ρ of $G := B_{4,4,4}/\Gamma_0$ verifies*

$$\sum_i \alpha_i \chi_{\bar{\rho}}(\theta_i r) \neq 0 \tag{1}$$

ρ appears as an irreducicble component of $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$. Here, α_i's are explicitely determined integers and $\bar{\rho}$ denotes the irreducible representation of semidirect product $G \rtimes \langle r\rangle$ which restricts to ρ, χ_* the character of the representation.

Since $B_{4,4,4}$ is known to be arithmetic lattice of $SO(3,1)$ over number field $K = \mathbb{Q}(\sqrt{5})$ (cf. [HLM1]) we can consider the congruence subgroups. For the case that Γ_0 is a principle congruence subgroup associated prime ideals of K we can compute the linear combination term on the left hand side of (1).

Theorem B. *Let $\Gamma_\mathfrak{p}$ be a principle congruence subgroup of $B_{4,4,4}$ associated to prime ideal $\mathfrak{p}$ in K. Set $G = B_{4,4,4}/\Gamma_\mathfrak{p}$.*
(i) If $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 1 \mod 8$ every nontrivial irreducible representaion of G appears in $H_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3, \mathbb{C})$.
(ii) Let Γ be a congurence subgroup of $B_{4,4,4}$. If the image of Γ in G doesnot contain noncentral normal subgroup the frist betti number of $\Gamma\backslash\mathbb{H}^3$ is positive.

The method of the computation implies somewhat general type of result.

Theorem C. *Let Γ be a maximal $r_1r_2r_3$-normal, but not maximal normal subgroup of finite index in $B_{4,4,4}$. Then any nontrivial $r_1r_2r_3$-invariant irreducible representation of $G = B_{4,4,4}/\Gamma$ is an irreducible component of $H_1(\Gamma\backslash\mathbb{H}^3,\ \mathbb{C})$.*

3. Universal group $B_{4,4,4}$ and cell decomposition. The orbifold $B_{4,4,4}\backslash\mathbb{H}^3$ can be given by the pasting of hyperbolic polyhedron R according to the pattern in Fig. 1. Polyhedron R is a hyperbolic regular dodecahedron with right edge angle. We denote by θ_X the elliptic element of order 4 which pastes the side X to side X'. $B_{4,4,4}\backslash\mathbb{H}^3$ has the natural cell decomposition induced from faces of R.

For normal subgroup $\Gamma \subset B_{4,4,4}$ we can equivriantly lift the cell decomposition $\Gamma\backslash\mathbb{H}^3$. We denote by $(\mathfrak{F}_i)_\Gamma$ the set of i-cells in the decomposition. Labeling the cells according to Fig. 2 we can explicitely describe the action of G on $(\mathfrak{F}_i)_\Gamma$ as follows.

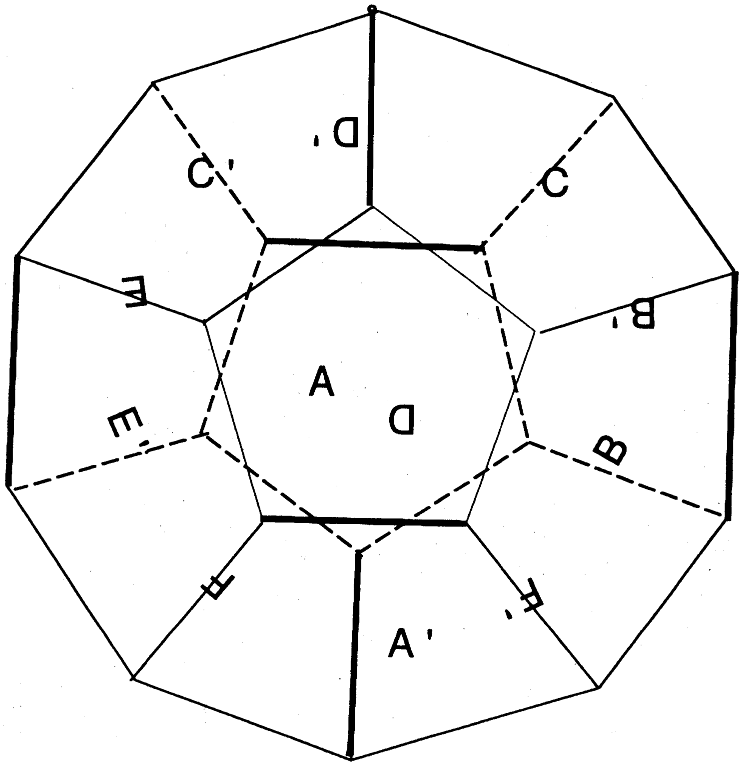

FIG. 1. Arrangement of Sides of Regular Dodecahedron R

Lemma 1. *Let Γ be a normal subgroup of $B_{4,4,4}$ and $G = B_{4,4,4}/\Gamma$. ($\simeq$ denotes the isomorphism as G-set.)*

(0) $(\mathfrak{F}_0)_\Gamma = G(\Gamma Q) \cup \{G(\Gamma P_x);\ x = a, b, ..f\}$,
$G(\Gamma Q) \simeq G, \quad G(\Gamma P_x) \simeq G/\langle\theta_X\rangle\ (x = a, b, .., f, X = A, B, .., F)$.

(1) $(\mathfrak{F}_1)_\Gamma = \{G(\Gamma xx');\ x = a, b, ..f\} \cup \{G(\Gamma y); y = ab, bc, ca, de, ef, fd\}$,
$G(\Gamma xx') \simeq G/\langle\theta_x\rangle\ (x = a, b, .., f), \quad G(\Gamma y) \simeq G\ (y = ab, bc, ca, de, ef, fd)$.

(2) $(\mathfrak{F}_2)_\Gamma = \{G(\Gamma X);\ X = A, B, .., F\}.G(\Gamma X) \simeq G\ (X = A, B, .., F)$

(3) $(\mathfrak{F}_3)_\Gamma = G(\Gamma R) \simeq G$.

The lemma gives us the description of G-chain complex $\{C_*, \partial\}$ associated to cell decompostion $(\mathfrak{F}_*)_\Gamma$ as follows.

$$C_0 \simeq \mathbb{C}[G] \cdot v_Q \oplus \bigoplus_x \mathbb{C}[G/\langle\theta_X\rangle] \cdot v_x := C_0' \oplus C_0''$$

$$C_1 = \bigoplus_x \mathbb{C}[G/\langle\theta_X\rangle] \cdot e_x \oplus \bigoplus_y \mathbb{C}[G] \cdot e_y := C_1' \oplus C_1''$$

$$C_2 = \bigoplus_x \mathbb{C}[G] \cdot s_X, \quad C_3 = \mathbb{C}[G] \cdot c_R.$$

where the summation indexes varies according to the description in Lemma 1 and v_*, e_*, s_*, and c_* are the oriented cells of the corresponding 0-,1-,2- and 3-cells. We also decompose C_0 and C_1 into two summands.

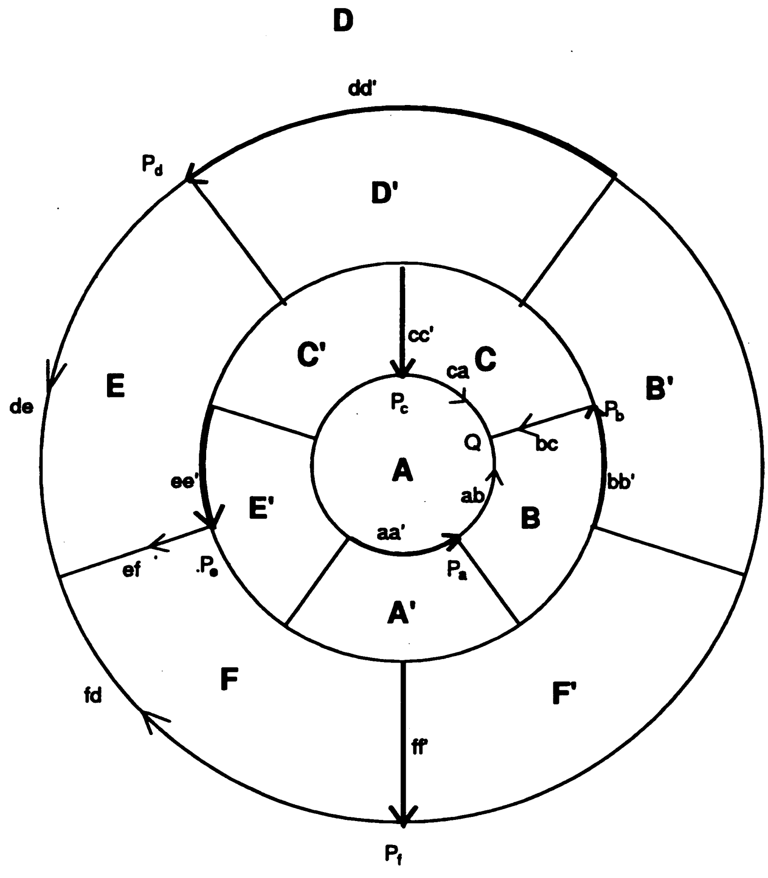

FIG. 2. Edges and Vertexes

In addition, if Γ is characterized by r $\{C_*, \partial\}$ has the action of $G \rtimes \langle r \rangle$. Observing the action of r in Fig. 1 we can explicitely describe action on the chain complex $\{C_*, \partial\}$ as follows.

Lemma 2. *Suppose Γ is r_1-normal. Then the action of r_1 on C_* is described as follows.*

$$
\begin{aligned}
&C_0' \ni \alpha \mapsto {}^{r_1}\alpha\theta_B \in C_0',\\
&C_0'' \ni (\alpha_A, \alpha_B, .., \alpha_F) \longmapsto\\
&\qquad\qquad ({}^{r_1}\alpha_A\theta_B, {}^{r_1}\alpha_B, {}^{r_1}\alpha_F\theta_A, {}^{r_1}\alpha_D\theta_E, {}^{r_1}\alpha_E, {}^{r_1}\alpha_C\theta_D) \in C_0'',\\
&C_1' \ni (\alpha_a, \alpha_b, .., \alpha_f) \mapsto (-{}^{r_1}\alpha_d, {}^{r_1}\alpha_b, -{}^{r_1}\alpha_c, -{}^{r_1}\alpha_a, {}^{r_1}\alpha_e, -{}^{r_1}\alpha_f) \in C_1',\\
&C_1'' \ni (\alpha_{ab}, \alpha_{bc}, \alpha_{ca}, \alpha_{de}, \alpha_{ef}, \alpha_{fd}) \longmapsto\\
&\qquad ({}^{r_1}\alpha_{ab}\theta_B, {}^{r_1}\alpha_{bc}\theta_B, {}^{r_1}\alpha_{fd}\theta_A\theta_C, {}^{r_1}\alpha_{de}\theta_E, {}^{r_1}\alpha_{ef}\theta_E, {}^{r_1}\alpha_{ca}\theta_D\theta_F) \in C_1'',\\
&C_2 \ni (\alpha_A, \alpha_B, .., \alpha_F) \longmapsto\\
&\qquad\qquad ({}^{r_1}\alpha_D\theta_A, {}^{r_1}\alpha_B\theta_B, -{}^{r_1}\alpha_C, {}^{r_1}\alpha_A\theta_D, {}^{r_1}\alpha_E\theta_E, -{}^{r_1}\alpha_F) \in C_2,\\
&C_3 \ni \alpha \mapsto -{}^{r_1}\alpha \in C_3.
\end{aligned}
$$

Moreover if ρ is a r_1-invariant irreducible representation of G these actions restrict to the homogeneous components of ρ.

Lemma 3. *Suppose Γ is $r_1r_2r_3$-normal. The actions of $r_1r_2r_3$ on six term modules C_0'', C_1', C_1'' and C_2 permute the components of pairs $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. The actions on C_0' and C_3 are given by*

$$C_0' \ni \alpha \mapsto {}^{r_1r_2r_3}\alpha\theta_E^{-1}\theta_F^{-1}\theta_A \in C_0', \quad C_3 \ni \alpha \mapsto -{}^{r_1r_2r_3}\alpha \in C_3.$$

If ρ is a $r_1r_2r_3$-invariant irreducible representation of G, these actions restrict to the homogeneous components of ρ.

4. General principle. Let $\Gamma \subset B_{4,4,4}$ be a r-normal subgroup of finite index and (C_*, ∂_*) the chain complex described in section 3. Then the complex is $G \rtimes \langle r \rangle$-module. The following two lemmas are direct consequences of elementary theory of representation of finite group and Poincare duality. Let $\mathrm{Irr}(G)$ denote the set of irreducible representation of G. For G-module M and $\rho \in \mathrm{Irr}(G)$ we denote by M_ρ the homogenious component of ρ.

Lemma 1. *For any $\bar\rho \in \mathrm{Irr}(G \rtimes S_0)$ chain complex (C_*, ∂_*) restricts to $G \rtimes S_0$-subcomplex $(C_{*,\bar\rho}, \partial_*|_{C_{*,\bar\rho}})$ and $H_*(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_{\bar\rho} \simeq H_*(C_{*,\bar\rho}, \partial_*|_{C_{*,\bar\rho}})$.*

Lemma 2. *For any $\rho \in \mathrm{Irr}(G)$, $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho \simeq H_2(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ as G-module.*

For $\rho \in \mathrm{Irr}(G)$ is r-invariant and M is a $G \rtimes \langle r \rangle$-module, r stabilizes homogeneous component M_ρ of G-module $\mathrm{Res}_G^{G\rtimes\langle r\rangle} M$. Hence M_ρ carries the action of $G \rtimes \langle r \rangle$ and we denote by $\bar{\mathcal{M}}_\rho$ the associated character of $G \rtimes \langle r \rangle$.

Proposition 1. *Suppose that Γ is r-normal and $\rho \in \mathrm{Irr}(G)$ is nontrivial and r-invariant. Then ρ is an irreducible component of $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$ if the generalized character*

$$\bar{\mathcal{E}}_\rho := \sum_i (-1)^i \bar{\mathcal{C}}_{i,\rho}$$

of $G \rtimes \langle r \rangle$ is not trivial.

Proof. Since $\Gamma\backslash\mathbb{H}^3$ is a connected closed 3-manifold, the characters of homologies $\mathcal{H}_0(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ and $\mathcal{H}_3(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ are trivial for $\rho \neq 1_G$. Hence the alternated sum $\bar{\mathcal{E}}_\rho$ is equal to the generalized character $\bar{\mathcal{H}}_2(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho - \bar{\mathcal{H}}_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ by Lemma 1. If the action of $G \rtimes \langle r \rangle$ induces the nontrivial character, either of $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ or $H_2(\Gamma\backslash\mathbb{H}^3, \mathbb{C})_\rho$ is at least nontrivial. The proposition follows from Lemma 2. Q.E.D.

5. Proof of Theorem A. In view of Proposition 1 $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$ has ρ as irreducible component if $\bar{\mathcal{E}}_\rho$ is nonzero. Thus Theorem A reduces to the computation of $\bar{\mathcal{E}}_\rho$. Let $\rho \in \mathrm{Irr}^r(G)$. For $\theta, g, h \in G$ with ${}^{hr}\theta \in \langle\theta\rangle$ we set

$$\varphi_\theta^r(g,h)_\rho : \ \mathbb{C}[G/\langle\theta\rangle]_\rho \ni \alpha \mapsto g\ {}^r\alpha h^{-1} \in \mathbb{C}[G/\langle\theta\rangle]_\rho, \ T_\theta^r(g,h)_\rho := \mathrm{Trace}(\varphi_\theta^r(g,h)_\rho).$$

We omit the upperscript r when it is obvious and the subscript θ if $\theta = 1$ (identity element).

Lemma 1. *Let r be either r_1 or $r_1r_2r_3$. Suppose Γ is r-normal and $\rho \in \mathrm{Irr}^r(G)$ is nontrivial. Then, for $r = r_1$*

$$\begin{aligned}\bar{\mathcal{E}}_\rho(gr_1) = &- T^{r_1}(g, \theta_E^{-1})_\rho - T^{r_1}(g,1)_\rho + T_{\theta_A}^{r_1}(g, \theta_B^{-1})_\rho \\ &+ T_{\theta_D}^{r_1}(g, \theta_E^{-1})_\rho + T_{\theta_C}^{r_1}(g,1)_\rho + T_{\theta_F}^{r_1}(g,1)_\rho.\end{aligned}$$

and for $r = r_1r_2r_3$,

$$\bar{\mathcal{E}}_\rho(gr_1r_2r_3) = T^{r_1r_2r_3}(g, \theta_A^{-1}\theta_F\theta_E)_\rho - T^{r_1r_2r_3}(g,1)_\rho.$$

Proof. The Lemma follows immediately from the definition of $\bar{\mathcal{E}}_\chi$ and the direct computation by the formulas in Lemma 2 in section 3. Q.E.D.

For simplicity we consider the case $r = r_1r_2r_3$, for which we only need the character formula for $T_\theta^r(*,*)$ with $\theta = 1$. The case $r = r_1$ is treated similarly but requires some more technical formula. First observe that the following is straightforward from the Cliford's theorem.

Lemma 2. *Let G be a finite group. Suppose that $r \in \mathrm{Aut}(G)$ is of order two and $\rho \in \mathrm{Irr}(G)$ is r-invariant. Then there exist exactly two irreducible representations $\bar{\rho}$ and $\sharp_r\bar{\rho}$ of $G \rtimes \langle r\rangle$ which restricts to ρ on G. The character of these satisfy $\chi_{\bar{\rho}}(x) + \chi_{\sharp_r\bar{\rho}'}(x) = 0$ for $x \in G \rtimes \langle r\rangle \setminus G$.*

It is immediately verified that the bi-action of $G \times G$ and the action of r on $\mathbb{C}[G]_\rho$ induces the action of the semidirect product $(G\times G) \rtimes \langle r\rangle$ given by the r-action $(g,h) \mapsto ({}^rg, {}^rh)$. We denote by σ the representation on $\mathbb{C}[G]_\rho$. Considering $(G\times G)\rtimes\langle r\rangle$ as a normal subgroup of $(G\rtimes\langle r\rangle)\times(G\rtimes\langle r\rangle)$ with index 2, we can define $\tau \in \mathrm{Irr}((G\times G)\rtimes\langle r\rangle)$ with $\mathrm{Res}^{(G\times G)\rtimes\langle r\rangle}_{G\times G}\,\tau = \rho\times\rho^*$ by

$$\tau := \mathrm{Res}^{(G\rtimes\langle r\rangle)\times(G\rtimes\langle r\rangle)}_{(G\times G)\rtimes\langle r\rangle}(\bar{\rho}\times\bar{\rho}^*).$$

Since $\mathbb{C}[G]_\rho$ is equivalent to $\rho\times\rho^* \in \mathrm{Irr}(G\times G)$, either $\sigma = \tau$ or $\sigma = \sharp\tau$ in view of Lemma 2. Thus $T^r(g,h) = \pm\chi_{\bar{\rho}}(gr)\chi^*_{\bar{\rho}}(hr)$. Therefor the computation of $T^r(g,h)$ reduces to the determination of the sign.

Lemma 3. $T^r(g,h) = \chi_{\bar{\rho}}(gr)\chi^*_{\bar{\rho}}(hr)$.

Proof. By the observations above we only have to prove that $\sigma \neq \sharp\tau$. Recall that $\mathbb{C}[G]_\rho$ is a simple component of $\mathbb{C}$-algebra $\mathbb{C}[G]$. Since the action of r induces a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[G]$ together with the conjugation by elements of G, the idempotent associated to r-invariant representation ρ is fixed by these actions. Hence we have

$$\langle \mathrm{Res}^{(G\times G)\rtimes\langle r\rangle}_H\,\sigma, 1_H\rangle_H \neq 0 \tag{5.1}$$

where H is the diagonal subgroup in $(G\rtimes\langle r\rangle)\times(G\rtimes\langle r\rangle)$, which is a subgroup of $(G\times G)\rtimes\langle r\rangle$. Since

$$\langle\bar{\rho},\bar{\rho}\rangle = \frac{1}{2|G|}\left\{\sum_{x\in G}|\chi_\rho(x)|^2 + \sum_{x\in G}|\chi_{\bar{\rho}}(xr)|^2\right\} = \frac{1}{2}\left\{\langle\rho,\rho\rangle + \frac{1}{|G|}\sum_{x\in G}|\chi_{\bar{\rho}}(xr)|^2\right\},$$

we have

$$1 = \frac{1}{|G|}\sum_{x\in G}|\chi_{\bar{\rho}}(xr)|^2. \tag{5.2}$$

By definition of $\sharp\tau$ and (5.2),

$$\begin{aligned}\langle \mathrm{Res}_H^{(G\times G)\rtimes\langle r\rangle}\sharp\tau, 1_H\rangle_H =& \frac{1}{|H|}\left\{\sum_{x\in G}|\chi_\rho(x)|^2 - \sum_{x\in G}|\chi_{\bar\rho}(xr)|^2\right\}\\ =& \frac{1}{2}\left\{\langle\rho,\rho\rangle_G - \frac{1}{|G|}\sum_{x\in G}|\chi_{\bar\rho}(xr)|^2\right\} = 0.\end{aligned}$$

Hence by (5.1) $\sigma \neq \sharp\tau$. Q.E.D.

As a consequence of Lemma 1 and Lemma 3, Theorem A follows for the case $r = r_1r_2r_3$. The explicite statement is as follows.

Theorem A. *Let Γ be a $r_1r_2r_3$-normal subgroup of finite index in $B_{4,4,4}$ and ρ a nontrivial $r_1r_2r_3$-invariant irreducible representation of G. If*

$$\chi_{\bar\rho}(\theta_A^{-1}\theta_F\theta_E r_1r_2r_3) + \chi_{\bar\rho}(r_1r_2r_3) \neq 0$$

for an irreducible representation $\bar\rho$ of $G \rtimes \langle r_1r_2r_3\rangle$ which restricts to ρ on G, ρ is an irreducible component of G-module $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$.

6. Remark on Theorem B and Theorem C. In view of Theorem A, Theorem B is the computation of character of $\bar\rho$ in the case that Γ is a congruence subgroup of arithmetic lattice. Since the character of irreducible representation of typical groups of Lie type is wellknown (cf. e.g. [Ca]) the problem reduces to the computation of the character of $\bar\rho$ from that of ρ. To clarify the points of the computations we briefly summarize the basic facts on arithemtic lattice and its congruence subgroups.
Let F be a field of characteristic$\neq 2$ and f a non-degenerate quadratic form on F^4. Set

$$O_f(F) := \{g \in \mathrm{GL}_4(F);\ g\cdot f = f\}$$

where g acts f by $g\cdot f(x,y) = f(g^{-1}x, g^{-1}y)$. For $\xi \in F^4$ with $f(\xi) \neq 0$ we denote by $r_\xi \in O_f(F)$ the orthogonal reflection with respect to plane $\xi^\perp$. r_ξ is obviously of determinant -1. Hence we have a normal subgroup of index two

$$SO_f(F) := \{g \in O_f(F);\ \det g = 1\}.$$

Spinorial norm Sp_f is the unique homomorphism of $O_f(F)$ to $F^*/(F^*)^2$ which takes reflection r_ξ to $f(\xi)$ mod $(F^*)^2$. Let $\Omega_f(F) = SO_f(F) \cap \mathrm{Ker}\, Sp_f$.

a. Arithmetic lattices Let k be a number field, $\mathfrak{o}$ the ring of integers in k and f a non-degenerate quadratic form on k^4. Set

$$O_f(\mathfrak{o}) := \{g \in O_f(k);\ \text{all entries of } g \text{ are integers}\}.$$

Suppose that v is a real infinite place of k and f induces quadratic form f_v at v of type (p,q). Then we have an associated embedding λ_v of $O_f(k)$ into $O(p,q:\mathbb{R})$. In particular, if $(p,q) = (3,1)$, $\mathrm{Ker}\, Sp_f$ and $\Omega_f(\mathfrak{o})$ embed into $\mathrm{Ker}\, Sp_{3,1}(\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^3)$ and $\Omega(3,1:\mathbb{R}) = \mathrm{Isom}_0(\mathbb{H}^3)$ respectively. The following is derived from the classical theorem due to Siegel.

Theorem (Siegel). *Suppose $k \neq \mathbb{Q}$ is a totally real number field and f is a non-degenerate anisotropic quadratic form on k^4. If f is definite at all infinite places except for v_0 and of type (3,1) at v_0,*

$$\Gamma_{v_0} := \lambda_{v_0}(O_f(\mathfrak{o})) \cap SO(3,1:\ \mathbb{R})$$

is a cocompact Kleinian group.

We say that Γ is an *arithmetic lattice* of $O(3,1:\mathbb{R})$ if Γ is commensurable with Γ_{v_0} in Theorem above. In [HLM1] Hilden, Lozano and Montesinos proved that $B_{4,4,4}$ is arithmetic lattice over $\mathbb{Q}(\sqrt{5})$

b. congruence subgroups For ideal $\mathfrak{m}$ of $\mathfrak{o}$ we define congruence subgroup $O_f(\mathfrak{m})$ by

$$O_f(\mathfrak{m}) := \{g \in O_f(\mathfrak{o}); g \equiv 1 \bmod \mathfrak{m}\}.$$

Clearly $O_f(\mathfrak{m})$ is an normal subgroup of $O_f(\mathfrak{o})$. Set

$$\Gamma'_{v_0} := \lambda_{v_0}(O_f(\mathfrak{o}) \cap \Omega_f(k)),\ \Gamma_{\mathfrak{m}} := \lambda_{v_0}(O_f(\mathfrak{m})) \cap \Omega_{f_0}(\mathbb{R}),$$
$$\Gamma'_{\mathfrak{m}} := \lambda_{v_0}(O_f(\mathfrak{m}) \cap \Omega_f(k)) := \Gamma'_{v_0} \cap \Gamma_{\mathfrak{m}}.$$

Note that Γ'_{v_0} and $\Gamma'_{\mathfrak{m}}$ are of finite index in Γ_{v_0} and $\Gamma_{\mathfrak{m}}$, respectively since those groups are finitely generated by its cocompactness and the spinorial norm maps those groups to the abelian group any non-trivial element of which are of order two. Suppose that $\mathfrak{p}$ is prime. Reducing the entries modulo $\mathfrak{p}$ we have injections

$$\iota_{\mathfrak{p}} : \Gamma_{v_0}/\Gamma_{\mathfrak{p}} \longrightarrow SO_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p}) \qquad \iota'_{\mathfrak{p}} : \Gamma'_{v_0}/\Gamma'_{\mathfrak{p}} \longrightarrow \Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$$

where $f_{\mathfrak{p}}$ denotes the quadratic form reduced from f modulo $\mathfrak{p}$. By Kneser's strong approximation $\iota'_{\mathfrak{p}}$ is surjective except for finite set $P_{B_{4,4,4}}$ of primes while $\iota_{\mathfrak{p}}$ may fail to be sujective by the lack of simply connectedness of SO_f. The following lemma is easily proved and describes when it fails.

Lemma. *Suppose $\mathfrak{p} \notin P_{B_{4,4,4}}$.*
(1) $\iota_{\mathfrak{p}}(B_{4,4,4}) = \Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 1 \mod 8$.
(2) $\iota_{\mathfrak{p}}(B_{4,4,4}) = SO_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 3 \mod 8$.

This dichotomy causes the restriction mod 8 in Theorem B (i). If $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 3 \mod 8$ we can prove that most of r-invariant characters apears in the first homology basing on Theorem A. Hence Theorem B (ii) follows from group theoretic technic and Proposition in Section 1.

We also have technically important dichotomy, which describes the two different types of group structures on $\Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$.

Lemma. (i) *Let d be a non-square element of $F_{\mathfrak{p}} := \mathfrak{o}/\mathfrak{p}$. Quadratic form $f_{\mathfrak{p}}$ belongs to the (unique) cogridient class of isotropic quadratic forms or that of anisotropic ones according to $-a$ is a square in $F_{\mathfrak{p}}$ or not.*
(ii) *If f is an isotropic quadratic form over $\mathbb{F}_q$, $\Omega_f(\mathbb{F}_q)$ is isomorphic to $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)/(\pm 1, \pm 1)$. If anisotropic, it is isomorphic to $SL_2(\mathbb{F}_{q^2})$.*

For the isotropic case the computation of the character of $\bar{\rho}$ is relatively easy by the direct product structure. Under the assumption of Theorem C the direct product structure is always the case (by the validity of Schreier's conjecture). On the other hand for the anisotropic case we have to develop a general theory to compute the character of $G \rtimes \langle r \rangle$ from that of G.

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